On Life Settlement Pricing

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Abstract

Although life settlements as financial products have been in existence and active use in financial markets for quite some time, their pricing has never reached the level of transparency and standardization envisioned by Wall Street. This lack of standardization has been and still is the major roadblock against widespread use of life settlements as investment, diversification and portfolio risk management tools. However, the recent crisis of 2007 has revealed high levels of correlation among existing financial instruments that are in widespread use. This revelation raised an avid interest in new financial instruments that show low correlation to strong market swings. In this respect, life settlements and related products such as death bonds have gained popularity among practitioners and academics alike. This paper proposes a standard pricing framework for life settlements that is consistent with existing methods of risk management and sensitivity analysis widely used in fixed income products.
I. Introduction

The revelation of high levels of correlation among existing financial instruments and their underlying assets during the mortgage-backed securities crisis in 2007 has sparked tremendous interest among investors for new instruments and assets, which have low, and potentially zero, correlation to the broader market. In this respect, Wall Street investment banks have turned their eyes towards one of the biggest asset classes in the United States: Life Insurance with $5.1 trillion in assets in 2007 (American Council of Life Insurers, 2008). When compared with the $5.08 trillion of Treasury securities in the outstanding US bond market debt in 2007 (Securities Industry and Financial Markets Association, 2009), the sheer size of the life insurance industry makes it a substantial untapped market. Moreover, Weber & Hause (2008) argue that a life insurance policy is uncorrelated against almost any other asset class as the payment of the death benefit is based on the event of death, not a market event that may cause a change in its value. These attributes seem to make life insurance an attractive investment vehicle and contribute significantly to the case for a secondary market in life insurance.

Therefore, the current idea of financial institutions involves the purchasing of large pools of life settlements - life insurance policies no longer needed or wanted by their owners - at prices higher than their cash surrender values, which are denoted as the buy back prices of the policies by the insurer. These pools are then securitized into bonds, or the so-called death bonds, that will be sold to investors following the promise of an attractive return, a greater yield to maturity than that offered by a same-maturity government bond without risk adjustment, and low correlation to the broader market (BusinessWeek, 2007). So far, most of the life settlement transactions have been done by life settlement providers, which have typically sold these policies to hedge funds. However, Wall Street firms now recognize the profit prospects in life settlement transactions and banks, including Goldman Sachs and Credit Suisse, have already acted well ahead of their rivals in preparing for a future secondary market in life insurance (New York Times, 2009). The Economist (2009) reports an estimated market size of $18 billion for life settlements as of June 2009 and Kamath & Sledge (2005) expect it to grow to a $160 billion industry by 2010. Despite increasing interest in the industry and the gradual expansion of the market, further growth, according to A.M. Best (2008), depends on increased clarity, standardization as well as transparency of the valuation of life settlements.

According to a research study by Insurance Studies Institute (2008), there are currently two main pricing methodologies: Deterministic Pricing and Prob-
abilistic Pricing. Since the value of the life settlement is a function of the life expectancy for the deterministic model and of the probability density function of death conditional on survival until the transaction time for the probabilistic model, a mortality forecasting model needs to be developed to analytically describe these actuarial concepts. Moreover, the value of the life settlement is dependent on the credit risk of the insurer as the secondary market purchaser of the policy needs to be compensated for its credit risk exposure. Consequently, the credit risk of the insurer needs to be priced into the appropriate valuation methodology.

This paper, hence, constructs an analytical valuation expression for life settlements that incorporates the probability density function of death conditional on survival until transaction time, the credit risk of the insurer and the term structure of interest rates. In reaching this result, it first conducts a literature review and states in the light of existing research its contribution to the field of life settlement pricing. It then lays out the theoretical framework, describes the data used, implements the theory to data, presents its results and concludes with a discussion of its shortcomings and potential improvements.

II. Literature Review

Pricing Methodologies

Although my research into the economic literature has failed to find an analytical pricing expression for life settlements, it is possible to construct these expressions from the descriptions of the deterministic and probabilistic pricing methodologies for life settlement valuation as detailed in an Insurance Studies Institute research brief (2008). The basic assumption of the deterministic pricing model is that in addition to the purchase price, the secondary market purchaser of the life insurance policy continues to pay the policy premiums, conventionally monthly, until the insured’s estimated time of death, which corresponds to the policy owner’s life expectancy. The value of the life settlement is then determined by computing the present value of the policy face value, also named the death benefit, less the total present value of all future premium payments until the estimated time of death. The appropriate discount rate applied to the cash stream, assuming zero credit risk for the insurance company, is the discount rate from the Treasury yield curve, since the risk-free return the investor would have otherwise realized is the corresponding yield from the yield curve.

Consider continuous time $t$ such that $t \in [t_0, \infty)$, whereby $t_0, t_1, t_2, \ldots, t_\infty, \ldots$ denote monthly separated discrete points along the continuous time line, when the premium payments are made. Then,
Let $\mathcal{D}^E$ denote the estimated time of death, or the life expectancy, and $\mathcal{D}_t^E$ the last year in which a premium payment is made, $V_{t_0}$ denote the $t_0$ value of the life insurance policy, $V_{\text{terminal}}$ the policy face value, or alternatively the death benefit that will be paid out upon the policy owner’s death, $A_t$, the monthly premium payment at time $t$, and $r_t$ the interest rate at time $t$.

**Definition** Assuming continuous compounding, the Deterministic Pricing Model is defined as,

$$V_{t_0} = V_{\text{terminal}} \cdot e^{-\int_{t_0}^{\mathcal{D}_t^E} r(\tau)d\tau} - \sum_{i=0}^{\mathcal{D}_t^E} A_t \cdot e^{-\int_{t_0}^{t_i} r(\tau)d\tau}$$

(1)

The deterministic pricing approach is built on the assumption of equivalence between the expected time of death, $\mathcal{D}^E$, and the actual time of death, $\mathcal{D}$. However, this assumption is a strong and unrealistic one as data on individual hazard curves indicate substantial probability of death occurrence before as well as after the estimated time of death.

Consequently, a probabilistic pricing methodology that takes into account individual hazard curves is more suitable for the valuation of life settlements. The probabilistic approach computes the probability density function of death conditional on survival until the transaction time to assign probabilistic weights to cash flows. In addition to the notation introduced above, let $pdf_D(t; C|\mathcal{D} \geq t_0)$ be the probability density function of time until death for a person born into a given cohort $C$ conditional on survival until $t_0$.

**Definition** Assuming continuous compounding, the Probabilistic Pricing Model is defined,

$$V_{t_0} = \int_{t_0}^{\infty} V_{\text{terminal}} \cdot pdf_D(\tau; C|\mathcal{D} \geq t_0) \cdot e^{-\int_{t_0}^{\tau} r_s ds} d\tau$$

$$- \sum_{i=0}^{t_1} e^{-\int_{t_0}^{t_i} r_s ds} \cdot A_{t_i} \cdot \int_{t_i}^{\infty} pdf_D(\tau; C|\mathcal{D} \geq t_0) d\tau$$

(2)

In the Probabilistic Pricing Model, both the cash inflow and outflow are weighted at each time step by the conditional probability density function of time until death and are discounted back to the present using the appropriate discount factor.
Mortality Forecasting Methods

With the selection of the appropriate pricing methodology, a mortality forecasting model is needed to describe the conditional probability density function of time until death. In this respect, Lee & Carter (1992) developed a commonly used extrapolative method using statistical time series techniques, in which the mortality level was represented by a single index. The logarithm of the central mortality rate, denoted $m_{x,t}$, is modeled as a linear function of unobserved period-specific intensity index, $k_t$, with parameters dependent on age, $a_x$ and $b_x$.

$$\ln(m_{x,t}) = a_x + b_x k_t + \epsilon_{x,t}$$  \hspace{1cm} (3) \\

where $m_{x,t}$ is defined as the percentage of deaths of people aged $x$ last birthday in an average population during calendar year $t$ (Plat, 2007). In the model, $a_x$ is the general shape of the hazard rate across different ages and $b_x$ indicates how different ages react to changes in $k_t$ such that,

$$\frac{d}{dt} \ln(m_{x,t}) = b_x \cdot \frac{d}{dt} k_t$$  \hspace{1cm} (5)

where $k_t$ is described by a random walk with constant drift.

Lee (2000) identifies that extrapolation may not always be a sensible approach to employ. He argues that the behavior of the mortality index, $k_t$, between 1900 and 1996, the time horizon used in the Lee & Carter model, is not representative of the historical trend and therefore cannot possibly reflect a fundamental property of the mortality change over time. He also points out the Lee-Carter model’s failure to incorporate positive future effects, such as breakthroughs in medical technology, that may accelerate mortality decline.

Renshaw & Haberman (2006) expand on the Lee-Carter model to incorporate cohort effects - effects dependent on year of birth,

$$\ln(m_{x,t}) = a_x + b^1_x \cdot k_t + b^2_x \cdot \gamma_{t-x}$$  \hspace{1cm} (6)

where $\gamma_{t-x}$ is the cohort effect. For countries with observed cohort effects in the past, Renshaw-Haberman model provides a much better fit than the Lee-Carter model.

Cairns et al (2006a) introduce a model of $q_{x,t}$, the initial mortality rate - defined as the probability that a person aged $x$ dies in the next calendar year (Plat, 2007) - with a bivariate ($k^1_t, k^2_t$) random walk with drift,

$$\text{logit}(q_{x,t}) = \ln\left(\frac{q_{x,t}}{1 - q_{x,t}}\right) = k^1_t + k^2_t(x - \bar{x})$$  \hspace{1cm} (7)
where $\bar{x}$ is the mean age. However, this model is designed for higher ages only, so it yields a poor fit for lower ages and Plat (2007) argues that it results in biologically unreasonable projections.

Further contributions to the field of stochastic mortality forecasting including Renshaw & Haberman 2005, Currie 2006 and Cairns et al 2008 allow for phenomena such as cohort effects and introduce further sources of randomness by describing stochastic processes for more parameters in the model for $\ln(m_x)$. Plat (2007), however, highlights that cohort effects are observed mostly until 1945 since these effects only materialize later in life. Therefore, application of mortality forecasting models with cohort effects results in the projection of the cohort effects for young ages, which can be volatile, into the future. Given the possibly temporary nature of these effects, such as drug and alcohol abuse, it is uncertain whether they are actually persistent into the future.

**Credit Risk and Credit Spread of a Risky Bond**

Quantification of credit risk is essential to accurate defaultable contingent claims analysis. In the case of life settlements, the secondary market purchaser of the life insurance policy is exposed to the default risk of the insurer. Hence, the purchaser of the life settlement needs to be compensated for the credit risk of the insurer with an additional spread over the risk-free yield. There are essentially two approaches to credit risk modeling: (1) Structural Models and (2) Reduced-Form Models. While structural models provide a link between the credit spread and the capital structure of a firm and thus provide an economic intuition for the credit risk, reduced-form models do not provide any such economic intuition and characterize default as a sudden change.

**Structural Model**

One popular structural method for the assessment of credit risk is the Merton model (Merton, 1974). This model is the first structural model that gives firm default an economic intuition and links the default event to underlying firm asset, debt and equity dynamics. Its simplicity, mathematical elegance and fundamental connection to the Black-Scholes equity option pricing framework have fueled Merton model’s widespread use. Current structural methods, those including endogenous default (Leland and Toft, 1996) or proposing jump-diffusion processes to describe the asset value process (Zhou, 1997), are all based on the original paper by Merton.

The model assumes that a firm has a certain amount of zero-coupon debt that will become due at a future time $T$. Thus, the company defaults if it fails to meet its payment obligation, that is, the value of its assets is less than the
promised debt payment at $T$. In this case, the equity, $E$, of the company becomes a European call option on the assets of the firm with maturity $T$ and strike price equal to the face value of the debt $D$. In the model, the assets, $A$, follow a log-normal diffusion process,

$$dA_t = \mu_A A_t dt + \sigma_A A_t dW$$  \hspace{1cm} (8)

where $\mu_A$ and $\sigma_A$, assumed constant, are the instantaneous rate of return and volatility of the assets respectively and $W$ is a standard Wiener process such that $dW \sim N(0, dt)$.

**Definition** The residual value after the debt repayment is,

$$E_T = \max(A_T - D, 0)$$  \hspace{1cm} (9)

Using the Black-Scholes option pricing formula and assuming constant interest rate $r$, the time $t$ value of the equity, $E_t$, is,

$$E_t = A_t \Phi(d_1) - D e^{-r(T-t)} \Phi(d_2)$$  \hspace{1cm} (10)

$$d_1 = \frac{\ln(A_t e^{r(T-t)}/D) + \frac{1}{2} \sigma_A \sqrt{T-t}}{\sigma_A \sqrt{T-t}} \hspace{0.5cm} d_2 = d_1 - \sigma_A \sqrt{T-t}$$  \hspace{1cm} (11)

Since, the market value of the firm’s assets, $A$, and the instantaneous volatility of assets, $\sigma_A$, are not directly observable, the model estimates $A_t$ and $\sigma_A$ from the market value of the firm’s equity and the equity’s instantaneous volatility using an approach suggested by Jones *et al* (1984). Since, the equity value is a function of the asset value, Ito’s Lemma can be used to show,

$$E_t \sigma_E = \frac{\partial E}{\partial A_t} \sigma_A$$  \hspace{1cm} (12)

Since $\frac{\partial E}{\partial A_t} = \Phi(d_1)$ from Equation (10),

$$[A_t \Phi(d_1) - D e^{-r(T-t)} \Phi(d_2)] \sigma_E = \Phi(d_1) A_t \sigma_A$$  \hspace{1cm} (13)

Then define $L_t$ to be a measure of the leverage of a firm at time $t$ such that,

$$L_t = \frac{D e^{-r(T-t)}}{A_t}$$  \hspace{1cm} (14)

It follows from Equation (13) that,

$$\sigma_E = \frac{\Phi(d_1) \sigma_A}{\Phi(d_1) - L(t) \Phi(d_2)}$$  \hspace{1cm} (15)

Equations (10) and (15) are solved for the two unknowns $A_t$ and $\sigma_A$. 

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To compute the credit spread on the defaultable bond, denoted $s$, we use an accounting identity.

**Definition** The value of a company’s assets at any point equals the sum of its equity value and debt. Let $B_t$ denote the market value of debt at time $t$ and $y$ the bond yield,

$$A_t = E_t + B_t$$  \hspace{1cm} (16)

It follows from Equation (10) that,

$$B_t = A_t - A_t \Phi(d_1) - D e^{-r(T-t)} \Phi(d_2) = A_t \Phi(-d_1) + L_t \Phi(d_2)$$  \hspace{1cm} (17)

Since the time $t$ market value of the debt is defined by its yield such that $B_t = D e^{-y(T-t)}$,

$$D e^{-y(T-t)} e^{-(r-y)(T-t)} = e^{-(r-y)(T-t)} A_t \Phi(-d_1) + L_t \Phi(d_2)$$  \hspace{1cm} (18)

$$D e^{-r(T-t)} = e^{-(r-y)(T-t)} A_t \Phi(-d_1) + L_t \Phi(d_2)$$  \hspace{1cm} (19)

$$(r - y)(T - t) = \ln \left( \frac{1}{L_t} \left[ \Phi(-d_1) + L_t \Phi(d_2) \right] \right) = \ln \left( \frac{\Phi(-d_1)}{L_t} + \Phi(d_2) \right)$$  \hspace{1cm} (20)

Thus,

$$s = y - r = -\ln \left( \frac{\Phi(-d_1)}{L_t} + \Phi(d_2) \right) \frac{1}{T - t}$$  \hspace{1cm} (21)

**Reduced-Form Model**

**Definition** The default time, $\mathbb{D}$, is the first jump of a Poisson process, so it is exponentially distributed with intensity $\theta$,

$$\mathbb{P}(\mathbb{D} > T) = e^{-\theta(T-t_0)}$$  \hspace{1cm} (22)

A defaultable claim of $1$ at time $T$ either pays $1$ at time $T$ with a probability $e^{-\theta(T-t_0)}$ or pays nothing due to default with probability $1 - e^{-\theta(T-t_0)}$. If we denote the time zero price of this defaultable claim by $P^c_{(t_0,T)}$, the interest rate by $r$ which is assumed constant and the risk-neutral probability measure by $\mathbb{Q}$,

$$P^c_{(t_0,T)} = e^{-r(T-t_0)} \mathbb{E}^Q(1_{\mathbb{D} > T})$$  \hspace{1cm} (23)

$$= e^{-r(T-t_0)} [1 \cdot \mathbb{P}^Q(\mathbb{D} > T) + 0 \cdot \mathbb{P}^Q(\mathbb{D} \leq T)] = e^{-r(T-t_0)} [1 \cdot e^{-\theta(T-t_0)}]$$  \hspace{1cm} (24)

$$= e^{-y(T-t_0)} = e^{-(r+\theta)(T-t_0)}$$  \hspace{1cm} (25)
Therefore, the effect of risk of default on the yield of a defaultable contingent claim is to add a spread of $\theta$.

III. Contribution of Financial Literature

My paper aims to construct a simplified mortality forecasting model, similar to the original Lee-Carter model, based on the analysis of $q_{x,t}$ with respect to $x$ for a given year $t = \tilde{t}$. Then, $q_{x,\tilde{t}}$ is a series of observations of the hazard rate across different age groups $x$ in a given year, $\tilde{t}$. Thus, $q_{x,\tilde{t}}$ is modeled as an exponential function with two parameters. Using nonlinear least squares regression technique, each parameter is estimated for all the given years to result in two separate time series, which are assumed independent of each other. Each time series is then modeled as a stochastic differential equation, where the drift is parametrized with a deterministic function and the diffusion is described by a mean-reverting Ornstein-Uhlenbeck process, which is the continuous time analog of the discrete time AR(1) process. Then, using the definition of the hazard rate as the ratio of the conditional probability density function of time until death to the survival distribution function, we derive from the hazard rate process, the conditional probability density function of time until death and the cumulative mortality distribution, which are used in the pricing model.

Additionally, my paper intends to investigate the standard assumption of zero correlation between life insurance instruments, specifically the life settlement, and the broader market. In contrast to the claim that “the death benefit is based on the event of death - not a market event” (Weber & Hause, 2008, pp. 66), the death benefit can only be collected if the insurer does not default. Therefore, the policyholder as well as the secondary market purchaser of the life insurance policy are exposed to considerable credit risk through the insurer (Cowley & Cummins, 2005), although almost all life insurance companies have above A rating. Consequently, in a life settlement transaction, the secondary market purchaser of the policy not only prices the conditional probability of death of the policyholder in the future but also the credit risk of the insurer. The inclusion of credit risk has two main consequences. First, an additional credit spread is priced into the discount factor. Second, assuming the stock market performance is an appropriate proxy for the market participants’ reactions to news, the value of the life settlement is positively correlated with the stock market performance through the credit risk of the insurer. Since almost all insurance companies invest in assets and instruments that are readily available in the market, the returns on these assets are linked to the performance of the market. Therefore, while it is not reasonable to expect a strong correlation between the credit spreads and the stock market performance on a daily basis,
in times of unexpectedly strong performances, either in the negative or the positive direction, by the stock market, credit spreads should in theory reflect a movement in the opposite direction as changes in the asset values might affect the companies’ ability to meet their debt repayment obligations.

I aim to quantify the credit risk exposure of the life settlement purchaser via the spot credit spread. The spot credit spread, also termed the Z-spread, is computed by discounting the cash flows of a corporate bond by the spot yield curve and pricing in an additional spread to match the observed bond price.

IV. Theoretical Framework

Mortality Forecasting Model

Modeling \( q_{x,t} \) for a given \( t = \bar{t} \) as an exponential function, let \( \beta_1^1 \) and \( \beta_2^2 \) \( \beta_1^2 \) be two parameters of the exponential hazard rate function \( q_{x,t} \)

\[
q_{x,t} = \beta_1^1 e^{\beta_2^2 x} \tag{26}
\]

where \( \beta_1^1 \) and \( \beta_2^2 \) are modeled as two stochastic differential equations whose long term trends are parameterized by deterministic functions of time and non-anticipative error terms described by two independent mean-reverting Ornstein-Uhlenbeck processes.

Definition Let \( W_t^\epsilon \) and \( W_t^\varphi \) be two independent Brownian motions such that \( dW_t^\epsilon, dW_t^\varphi \sim N(0, dt) \), \( \epsilon \) and \( \varphi \) the errors of the diffusion processes, \( g_{\beta_1}(t) \) and \( g_{\beta_2}(t) \) the deterministic functions of \( t \), \( \vartheta \) and \( \varphi \) the mean-revision rates \( \sigma_\epsilon, \sigma_\varphi \), and \( \mu_\epsilon, \mu_\varphi \) the standard deviations and the means of errors for \( \beta_1^1 \) and \( \beta_2^2 \) respectively,

\[
d\beta_1^1 = dg_{\beta_1}(t) + \vartheta(\mu_\epsilon - \epsilon_t)dt + \sigma_\epsilon dW_t^\epsilon \tag{27}
\]

\[
d\beta_2^2 = dg_{\beta_2}(t) + \varphi(\mu_\varphi - \varphi_t)dt + \sigma_\varphi dW_t^\varphi \tag{28}
\]

For some arbitrary initial point in time, \( o \), for which the initial values, \( \beta_1^1, g_{\beta_1}(o) \) and \( \epsilon_\varphi \) are known,

\[
\int_o^t d\beta_1^1 = \int_o^t dg_{\beta_1}(\tau) + \int_o^t \vartheta(\mu_\epsilon - \epsilon_\tau) d\tau + \int_o^t \sigma_\epsilon dW_\tau^\epsilon \tag{29}
\]

\[
\beta_1^1 = \beta_1^1 + g_{\beta_1}(t) - g_{\beta_1}(o) + \epsilon_\varphi e^{-\varphi(t-o)} + \mu_\epsilon(1 - e^{-\varphi(t-o)}) + \int_o^t \sigma_\epsilon e^{-\varphi(t-\tau)} dW_\tau^\epsilon \tag{30}
\]

and similarly,

\[
\beta_2^2 = \beta_2^2 + g_{\beta_2}(t) - g_{\beta_2}(o) + \epsilon_\varphi e^{-\varphi(t-o)} + \mu_\varphi(1 - e^{-\varphi(t-o)}) + \int_o^t \sigma_\varphi e^{-\varphi(t-\tau)} dW_\tau^\varphi \tag{31}
\]
Since both $dW_t^x$ and $dW_t^z$ are normally distributed, the $\beta_t^1$ and $\beta_t^2$ processes are also normally distributed. Therefore, the exact distributions of the $\beta_t^1$ and $\beta_t^2$ processes are completely described by their expectation and variance,

$$\mathbb{E}(\beta_t^1) = \beta_0^1 + g_{\beta_1}(t) - g_{\beta_1}(\theta) + \epsilon_\sigma e^{-\theta(t-\theta)} + \mu_\epsilon(1 - e^{-\theta(t-\theta)})$$

(32)

$$Var(\beta_t^1) = Var\left(\int_0^t \sigma_\epsilon e^{-\theta(t-\tau)}dW_t^z\right)$$

(33)

**Lemma 4.1** Variance is defined as,

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

(34)

**Lemma 4.2** The mean and the variance of the stochastic integral $\int \theta_t dW_s$ are

$$\mathbb{E}(\int_0^t \theta_s dW_s) = 0$$

(35)

$$\mathbb{E}(\int_0^t \theta_s dW_s)^2 = \mathbb{E}(\int_0^t \theta_s^2 ds)$$

(36)

It follows from Lemma 4.1 and Lemma 4.2 that,

$$\mathbb{E}\left(\int_0^t \sigma_\epsilon e^{-\theta(t-\tau)}dW_t^z\right) = 0$$

(37)

$$Var\left(\int_0^t \sigma_\epsilon e^{-\theta(t-\tau)}dW_t^z\right) = \mathbb{E}\left\{\left(\int_0^t \sigma_\epsilon e^{-\theta(t-\tau)}dW_t^z\right)^2\right\}$$

(38)

$$= \mathbb{E}\left(\int_0^t \sigma_\epsilon^2 e^{-2\theta(t-\tau)}d\tau\right) = \sigma_\epsilon^2 e^{-2\theta t} - e^{-2\theta \theta}$$

(39)

$$= \sigma_\epsilon^2 \left(1 - \frac{e^{-2\theta(t-\theta)}}{2\theta}\right)$$

(40)

Thus, $\beta_t^1$ is distributed normally,

$$\beta_t^1 \sim N(\beta_0^1 + g_{\beta_1}(t) - g_{\beta_1}(\theta) + \epsilon_\sigma e^{-\theta(t-\theta)} + \mu_\epsilon(1 - e^{-\theta(t-\theta)}), \sigma_\epsilon^2 \left(1 - \frac{e^{-2\theta(t-\theta)}}{2\theta}\right))$$

**Definition** Let $X$ be a random variable such that $X \sim N(0, 1),$

$$\beta_t^1 = \beta_0^1 + g_{\beta_1}(t) - g_{\beta_1}(\theta) + \epsilon_\sigma e^{-\theta(t-\theta)} + \mu_\epsilon(1 - e^{-\theta(t-\theta)}) + \sigma_\epsilon \sqrt{\frac{1 - e^{-2\theta(t-\theta)}}{2\theta}} X$$

(41)

with
Similarly, let \( Y \) be a random variable such that \( Y \sim N(0,1) \), then

\[
\beta^1_t = \beta^1_o + g_{\beta_1}(t) - g_{\beta_1}(o) + \epsilon e^{-\theta(t-o)} + \mu e(1 - e^{-\theta(t-o)})
\]  

(42)

\[
\text{Var}(\beta^1_t) = \sigma^2 e^{-2\theta(t-o)}
\]  

(43)

Lemma 4.3 Let \( C \) denote the birth time of the individual. Since age is the difference between the current time and the time of birth,

\[
x = t - C
\]  

(45)

where \( C \) is fixed for any given individual.

Lemma 4.4 The hazard rate is defined as,

\[
q_{t:C} = \frac{pdf_D(t;C)}{1 - cdf_D(t;C)}
\]  

(46)

Proposition 4.5 Let \( H(t;C) = 1 - cdf_D(t;C) \) and \( \frac{\partial}{\partial t} H(t;C) = h(t;C) = -pdf_D(t;C) \). It follows from Lemma 4.3 and Lemma 4.4 that,

\[
q_{t:C} = \frac{-h(t;C)}{H(t;C)}
\]  

(47)

Integrating both sides from \( C \) to \( t \),

\[
- \int_C^t q_{t:C} d\tau = \ln(H(t;C))
\]  

(48)

Definition The probability density function of death and cumulative probability distribution of death are defined as,

\[
cdf_D(t;C) = 1 - e^{-\int_C^t q_{t:C} d\tau}
\]  

(49)

\[
pdf_D(t;C) = \frac{\partial}{\partial t} \left[ 1 - e^{-\int_C^t q_{t:C} d\tau} \right]
\]  

(50)

where,
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or equivalently,

\[ q(t) = \{ \beta_0 + g_\beta(t) + g_\beta(o) + \epsilon \sigma e^{-\theta(t-t_0)} \} + \int_{t_0}^{t} \sigma \epsilon e^{-\theta(t-t_0)} dW_z \] (51)

\[ e^{(t-C) [\beta_0 + g_\beta(t) + g_\beta(o) + \epsilon \sigma e^{-\theta(t-t_0)} + \mu_\epsilon(1-e^{-\theta(t-t_0)}) + \int_{t_0}^{t} \sigma \epsilon e^{-\theta(t-t_0)} dW_z] \] (52)

where \( X, Y \sim N(0, 1) \) and \( X, Y \) are independent.

**Spot Z-spread**

Using the probabilistic pricing framework introduced previously, we now introduce credit risk into the model. From the perspective of the bank, the credit risk of the insurance company can be quantified with a static spread above the risk-free zero yield curve. The Z-spread is that static spread above the risk-free zero yield curve, which makes the sum of the present value of all the cash flows of the bond equal to its observed market price. In this sense, it is a measure of the company’s default and liquidity risk.

**Definition** Let \( r(t) \) be the interest rate at time \( t \), \( R(t, t) \) be the discount factor for a risk-free zero coupon bond maturing at time \( t \) as seen from \( t_0 \) and \( S_t \) be the Z-spread at time \( t_0 \). The price of a continuously compounded risk-free zero coupon bond paying 1$ at maturity is,

\[ P_{(t_0, t)} = e^{-\int_{t_0}^{t} r_s ds} \] (53)

\[ e^{-R(t_0, t)(t-t_0)} \] (54)

The price of an identical defaultable bond,

\[ P_{(t_0, t)} = e^{-\int_{t_0}^{t} r_s ds - S_t(t-t_0)} \] (55)

\[ e^{-[R(t_0, t)+S_t(t-t_0)]} \] (56)

Consider continuous time \( t \) such that \( t \in [t_0, t_n] \) with discretely separated time points \( t_0 < t_1 < t_2 < \ldots < t_i < t_{i+1} < \ldots < t_n \). Let \( C_{t_i} \) denote the coupon payment at time \( t_i \) and \( F \) the face value of the bond.

**Lemma 4.6** Each coupon payment, \( C_{t_i} \), of the coupon bond can be treated as a zero coupon bond with maturity equal to the coupon payment date, \( t_i \) and face value equal to the coupon itself.
It follows from Lemma 4.6 that,

\[ P_{(t_0, t_n)} = \sum_{i=1}^{n} C_i e^{-[R(t_0, t_i) + S_{t_0}](t_i - t_0)} + F e^{-[R(t_0, t_n) + S_{t_0}](t_n - t_0)} \]  

(57)

The spot Z-spread is the value of \( S_{t_0} \) that makes the right hand side of Equation (57) equal to the observed bond price \( P_{(t_0, t_n)} \)

**The Life Settlement Pricing Model**

Assumption 4.7 Assume that the only available life insurance contract available in the market is the continuous whole life insurance policy, although in real life there is a wide array of life insurance contracts with different cash flow dynamics.

**Definition** Combining the models and the assumption given above, the life settlement price is defined as the sum of the present values of all cash flows weighted by the conditional probability density function of time until death for an individual born into a cohort \( C \).

\[
V_{t_0} = \int_{t_0}^{\infty} V_{\text{terminal}} \cdot e^{-[R(t_0, \tau) + S_{t_0}](\tau - t_0)} \cdot \text{pdf}_D(\tau; C|D \geq t_0) \, d\tau 
\]

\[-\sum_{j=0}^{\infty} A_j \cdot e^{-[R(t_0, t_j) + S_{t_0}](t_j - t_0)} \cdot \int_{t_j}^{\infty} \text{pdf}_D(\tau; C|D \geq t_0) \, d\tau \]

(58)

**Lemma 4.8** From the definition of probability density function, for any \( t_0 > C \) and \( t \geq 0 \),

\[
\text{pdf}_D(t_0 + t; C|D \geq t_0) = \mathbb{P}(t + t_0 \leq D \leq t + t_0 + dt; C|D \geq t_0)
\]

for some infinitesimally small change \( dt \). Then, from the definition of conditional probability,

\[
\mathbb{P}(t + t_0 \leq D \leq t + t_0 + dt; C|D \geq t_0) = \frac{\mathbb{P}(t + t_0 \leq D \leq t + t_0 + dt \cap D \geq t_0; C)}{\mathbb{P}(D \geq t_0; C)}
\]

Let \( \omega_1 \) be the event that \( \{ t + t_0 \leq D \leq t + t_0 + dt \} \) and \( \omega_2 \) be the event that \( \{ D \geq t_0 \} \). Since \( \omega_1 \subseteq \omega_2 \) and \( \omega_1 \cap \omega_2 = \omega_1 \). Therefore,

\[
\frac{\mathbb{P}(t + t_0 \leq D \leq t + t_0 + dt \cap D \geq t_0; C)}{\mathbb{P}(D \geq t_0; C)} = \frac{\mathbb{P}(t + t_0 \leq D \leq t + t_0 + dt; C)}{\mathbb{P}(D \geq t_0; C)}
\]

which then results in,

\[
\text{pdf}_D(t_0 + t; C|D \geq t_0) = \frac{\text{pdf}_D(t_0 + t; C)}{1 - \text{cdf}_D(t_0; C)}
\]
It follows from Lemma 4.7 that for any $C \geq t_0 \geq t$, $pdf_D(t; C|D \geq t_0)$ is defined as,

$$pdf_D(t; C|D \geq t_0) = \frac{pdf_D(t; C) \cdot e^{-\int_{t_0}^{t} q_{t, c} \, dt}}{1 - cdf_D(t_0; C)}$$

(59)

Similarly, for any $C \leq t_0 \leq t$, $cdf_D(t; C|D \geq t_0)$ is defined as following,

$$cdf_D(t; C|D \geq t_0) = \frac{P(D \leq t; C|D \geq t_0)}{P(D \geq t_0; C)} = \frac{P(t_0 \leq D \leq t; C)}{P(D \geq t_0; C)}$$

(60)

Thus, we rewrite the pricing model as,

$$V_{t_0} = \int_{t_0}^{\infty} V_{\text{terminal}} \cdot e^{-[R(t_0, \tau) + S_{t_0}](\tau - t_0)} \cdot pdf_D(\tau; C|D \geq t_0) \, d\tau$$

(61)

$$- \sum_{i=0}^{\infty} A_{t_i} \cdot e^{-[R(t_0, t_i) + S_{t_0}](t_i - t_0)} \cdot \int_{t_i}^{\infty} pdf_D(\tau; C|D \geq t_0) \, d\tau$$

(62)

$$= \int_{t_0}^{\infty} V_{\text{terminal}} \cdot e^{-[R(t_0, \tau) + S_{t_0}](\tau - t_0)} \cdot q_{\tau, C} \cdot e^{-\int_{t_0}^{\tau} q_{s, c} \, ds} \, d\tau$$

(63)

$$- \sum_{i=0}^{\infty} A_{t_i} \cdot e^{-[R(t_0, t_i) + S_{t_0}](t_i - t_0)} \cdot e^{-\int_{t_0}^{t_i} q_{\tau, c} \, d\tau}$$

(64)

The hazard rate, $q_{t, C}$, is defined as,

$$q_{t, C} = \left\{ \beta_0^1 + g_{\beta_1}(t) - g_{\beta_1}(o) + \epsilon_\sigma e^{-\alpha(t-o)} + \mu_c(1 - e^{-\alpha(t-o)}) + \sigma_Y \sqrt{\frac{1 - e^{-2\alpha(t-o)}}{2\sigma^2}} X \right\}$$

(65)

$$e^{(t-C)\beta_0^1 + g_{\beta_1}(t) - g_{\beta_1}(o) + \epsilon_\sigma e^{-\alpha(t-o)} + \mu_c(1 - e^{-\alpha(t-o)}) + \sigma_Y \sqrt{\frac{1 - e^{-2\alpha(t-o)}}{2\sigma^2}} Y}$$

(66)

$X, Y \sim N(0, 1)$ and $X, Y$ are independent.

V. Data

This paper uses three main sources for data. The mortality data comes from The Human Mortality Database, which contains calculations of death rates and life tables for national populations of more than thirty countries, including the United States, Canada, Australia and the countries of western Europe. We use yearly life tables for the United States covering the years from 1933 until 2006 to model the mortality distributions. Each life table in the da-
Database contains mortality information for every age until 110. Between every two successive ages, the life table includes the central death rate, the probability of death, the number of deaths, the average length of survival for people dying in the interval and the number of survivors at the beginning of the interval.

The yield curve data comes from the U.S. Treasury’s website, which contains the yields on 1-month, 3-month, 6-month, 1-year, 2-year, 3-year, 5-year, 7-year, 10-year, 20-year and 30-year Treasury bonds from 1990 until today on a daily basis. We then fit a Nelson-Siegel function to the observed yields to construct the yield curve.

Finally, the data on the observed bond prices of the insurance company comes from the TRACE (Trade Reporting and Compliance Engine) database. The database contains information on corporate bond transactions of all brokers and dealers who are FINRA (Financial Industry Regulatory Authority) members. The database provides the transaction date, the transaction time, the effective yield to maturity and the price for the selected bond within the chosen time horizon.

VI. Implementation to Data

Mortality Forecasting Model

If we assume that people under the age of thirty are not eligible to enter into a life settlement transaction, it suffices to model hazard rates only for individuals of thirty years or more of age. For a given year \( t = \bar{t} \), we model the hazard rate \( q_{t,c} \) as an exponential function,

\[
q_{t,c} = \beta_1 e^{t} e^{-Bet} (t-c) \tag{68}
\]

where \( \beta_1 \) and \( \beta_2 \) are two time dependent parameters and can be estimated for a given year by non-linear least squares regression. For \( t = 2006 \), the plot of \( q_{2006,c} \) yields,
Estimating the two parameters $\beta^1_t$ and $\beta^2_t$ for all the years in which hazard rate observations are available results in a collection of random variables over time $\{\beta^1_t, \beta^2_t : t \in [0, \infty)\}$, which can be described by two stochastic differential equations. The plot of $\{\beta^1_t\}$ against $t$ yields,
The plot shows random mean-reverting fluctuations around a long term deterministic trend.

**Assumption 6.1** We model the change in $\beta_t^1$ as a change in a deterministic function of time with errors following a mean-reverting Ornstein-Uhlenbeck process,

$$d\beta_t^1 = \vartheta(\mu_i - \epsilon_i)dt + \sigma_i d\xi_t^i$$  \hspace{1cm} (69)

The solution to this stochastic differential equation is,

$$\beta_t^1 = \beta_0^1 + g_{\beta_1}(t) - g_{\beta_1}(0) + \epsilon_0 e^{-\vartheta(t-\mu_i)} + \mu_i (1 - e^{-\vartheta(t-\mu_i)}) + \int_0^t \sigma_i e^{-\vartheta(t-r)} d\xi_t^i$$  \hspace{1cm} (70)

**Assumption 6.2** The deterministic part of the function is modeled as an exponential function,

$$g_{\beta_1}(t) = \gamma_1 e^{\gamma_2 t}$$  \hspace{1cm} (71)

where $\gamma_1 = 0.0021$ and $\gamma_2 = -0.0271$ by non-linear least squares regression.

**Assumption 6.3** The residuals are modeled as an Ornstein-Uhlenbeck process,
\[ \epsilon_{t+dt} = \epsilon_t e^{-\vartheta t} + \mu_\epsilon (1 - e^{-\vartheta dt}) + \sigma_\epsilon \sqrt{\frac{1 - e^{-2\vartheta dt}}{2\vartheta}} X \] (72)

where \( X \sim N(0, 1) \).

The three parameters, \( \vartheta \) the mean-reversion rate, \( \mu_\epsilon \) the mean error value and \( \sigma_\epsilon \) the standard deviation of error terms are estimated using the maximum likelihood estimation methodology. After much algebraic manipulation, details of which are given in the appendix, the following solutions are achieved for \( \mu_\epsilon, \sigma_\epsilon \) and \( \vartheta \):

\[
\mu_\epsilon = \frac{\sum_{i=1}^{n} \epsilon_{t_i} \sum_{i=1}^{n} \epsilon_{t_i}^2 - \sum_{i=1}^{n} \epsilon_{t_{i-1}} \sum_{i=1}^{n} \epsilon_{t_i} \epsilon_{t_{i-1}}}{n(\sum_{i=1}^{n} \epsilon_{t_i}^2 - \sum_{i=1}^{n} \epsilon_{t_{i-1}} \epsilon_{t_i}) - [(\sum_{i=1}^{n} \epsilon_{t_{i-1}})^2 - \sum_{i=1}^{n} \epsilon_{t{i-1}} \sum_{i=1}^{n} \epsilon_{t_i}]} \] (73)

\[
\sigma_\epsilon = \frac{2\vartheta}{n(1 - e^{-2\vartheta dt})} \left[ \sum_{i=1}^{n} \epsilon_{t_i}^2 - 2e^{-\vartheta dt} \sum_{i=1}^{n} \epsilon_{t_i} \epsilon_{t_{i-1}} + e^{-2\vartheta dt} \sum_{i=1}^{n} \epsilon_{t_{i-1}}^2 \right] \] (74)

\[
\vartheta = -\frac{1}{dt} \ln \left\{ \frac{\sum_{i=1}^{n} \epsilon_{t_{i-1}} \epsilon_{t_i} - \mu_\epsilon \sum_{i=1}^{n} \epsilon_{t_{i-1}} - \mu_\epsilon \sum_{i=1}^{n} \epsilon_{t_i} + n\mu_\epsilon^2}{\sum_{i=1}^{n} \epsilon_{t_{i-1}}^2 - 2\mu_\epsilon \sum_{i=1}^{n} \epsilon_{t_{i-1}} + n\mu_\epsilon^2} \right\} \] (75)

This maximum likelihood estimators for \( \mu_\epsilon, \sigma_\epsilon \) and \( \vartheta \) are \(-8.0322 \cdot 10^{-5}\), \(5.1223 \cdot 10^{-4}\) and \(0.0244\) respectively. Thus,

\[
\beta_2^t = \gamma_1 e^{\vartheta t} (e^t - e^\vartheta) + \epsilon_\sigma e^{-\vartheta (t-\vartheta)} + \mu_\epsilon (1 - e^{-\vartheta (t-\vartheta)}) + \sigma_\epsilon \sqrt{\frac{1 - e^{-2\vartheta (t-\vartheta)}}{2\vartheta}} X \] (76)

where \( X \sim N(0, 1) \). Similarly, the plot of \( \{\beta_2(t)\} \) against \( t \) yields,
Assumption 6.4 We model the change in $\beta_2 t$ as a change in a deterministic function of time with errors following a mean-reverting Ornstein-Uhlenbeck process,

$$d\beta_2^2 = dg_{\beta_2}(t) + \varphi(\mu_\epsilon - \epsilon_t)dt + \sigma_\epsilon dW_\epsilon^\varphi$$

(77)

The solution of this stochastic differential equation is,

$$\beta_2^2 = \beta_0^2 + g_{\beta_2}(t) - g_{\beta_2}(o) + \epsilon_0 e^{-\varphi(t-o)} + \mu_\epsilon (1 - e^{-\varphi(t-o)}) + \int_o^t \sigma_\epsilon e^{-\varphi(t-\tau)}dW_\epsilon^\varphi$$

(78)

Assumption 6.5 The deterministic part of the function has the form,

$$g_{\beta_2}(t) = \alpha_1 + \alpha_2 t$$

(79)

where $\alpha_1 = 0.0505$ and $\alpha_2 = 0.0003$ by ordinary least squares regression.

Assumption 6.6 The residuals are then modeled as a mean-reverting Ornstein-Uhlenbeck process.
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where $Y \sim N(0, 1)$.

Using maximum likelihood estimation, the three parameters $\mu, \sigma, \varphi$ are $1.2120, 2.6502 \cdot 10^{-4}$ and $2.3015 \cdot 10^{-4}$. Thus,

$$
\beta_t^2 = \alpha_2(t - o) + \varepsilon_2 e^{-\varepsilon^2(t-o)} + \mu_2(1 - e^{-\varphi(t-o)}) + \sigma_2 \sqrt{\frac{1 - e^{-2\varphi(t-o)}}{2\varphi}} Y
$$

where $Y \sim N(0, 1)$.

**Definition** The hazard rate model is,

$$
q_{t|c} = \beta_1 c^{\beta_2(t-c)}
$$

Substituting in the expressions for $\beta_1$ and $\beta_2$,

$$
q_{t|c} = \left( \gamma_1 e^{\gamma_2(c^\mu - c^n)} + \sigma_1 e^{-\sigma_2(t-o)} + \mu_1(1 - e^{-\varphi(t-o)}) + \sigma_1 \sqrt{\frac{1 - e^{-2\varphi(t-o)}}{2\varphi}} Y \right)
$$

where $\gamma_1 = 0.0021, \gamma_2 = -0.0271, \mu = -8.0322 \cdot 10^{-5}, \sigma = 5.1223 \cdot 10^{-5}, \varphi = 0.0244, \alpha_1 = 0.0505, \alpha_2 = 0.0003, \mu = 1.2120, \sigma = 2.6502 \cdot 10^{-4}$ and $\varphi = 2.3015 \cdot 10^{-4}$.

The unconditional probability density function of time until death, $pdf_D(t; C)$ and the cumulative probability distribution of death, $cdf_D(t; C)$, are defined as,

$$
pdf_D(t; C) = q_{t|c} \cdot e^{-\int_c^t q_{t|c} d\tau}
$$

$$
cdf_D(t; C) = 1 - e^{-\int_c^t q_{t|c} d\tau}
$$

The simulation of the $pdf_D(t; C)$ against $t$ for the 1970 cohort, $C = 1970$ yields,
Since $q_{t:C}$ is a function of time $t$ with two normally distributed random variables $X, Y$, we can compute both its expected value and its variance. We therefore know the entire distribution of $q_{t:C}$ at any $t$ for any given cohort $C$. Although there is an entire distribution of the $q_{t:C}$, the best estimate of the actual distribution of $q_{t:C}$ is its expected value at any $t$.

**Definition** Let,

$$u(t) = \gamma_1 e^{\gamma_2 (e^t - e^0)} + \epsilon_0 e^{-\theta(t-t_0)} + \mu_1 (1 - e^{-\theta(t-t_0)}) + \sigma_1 \sqrt{\frac{1 - e^{-2\theta(t-t_0)}}{2\theta}} X$$ \hspace{1cm} (86)

$$v(t, C) = e^{(t-C)\alpha_2 (t-t_0) + \epsilon_0 e^{-\theta(t-t_0)} + \mu_1 (1 - e^{-\theta(t-t_0)}) + \sigma_1 \sqrt{\frac{1 - e^{-2\theta(t-t_0)}}{2\theta}} Y}$$ \hspace{1cm} (87)

**Lemma 6.7** For two independent random variables $A$ and $B$,

$$E(f(A)g(B)) = E(f(A))E(g(B))$$ \hspace{1cm} (88)

Since $X, Y$ are independent, it follows from Lemma 6.7 that,

$$E(q_{t:C}) = E(u(t)v(t, C)) = E(u(t))E(v(t, C))$$ \hspace{1cm} (89)

where,
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\[ \mathbb{E}(u(t)) = \gamma_1 e^{\gamma_2 (t - e^0)} + \epsilon_o e^{-\theta(t-o)} + \mu_e (1 - e^{-\theta(t-o)}) \] (90)

\[ \mathbb{E}(v(t, C)) = e^{(t-C)\left(\alpha_2(t-o)+\epsilon_o e^{-\phi(t-o)}+\mu_e (1-e^{-\phi(t-o)})\right)+\frac{1}{2}(t-C)^2\sigma^2_e 1-e^{-2\phi(t-o)}} \] (91)

Therefore,

\[ \mathbb{E}(q(t, C)) = \left[ \gamma_1 e^{\gamma_2 (t - e^0)} + \epsilon_o e^{-\phi(t-o)} + \mu_e (1 - e^{-\phi(t-o)}) \right] 
\cdot e^{(t-C)\left(\alpha_2(t-o)+\epsilon_o e^{-\phi(t-o)}+\mu_e (1-e^{-\phi(t-o)})\right)+\frac{1}{2}(t-C)^2\sigma^2_e 1-e^{-2\phi(t-o)}} \] (92)

The plot of pdf\(_D(t; C)\) based on \(\mathbb{E}(q(t, C))\) against \(t\) for the 1970 cohort, \(C = 1970\) yields,

Thus, we can now compute the mortality density function of any individual belonging to a certain cohort \(C\) to be used in the pricing model.

**VII. Results**

In this section, we first explore the distribution of life settlement prices with respect to the underlying stochastic distributions of the hazard function, the probability density function and the cumulative distribution function of death. Secondly, we analyze the sensitivity of life settlements to changes in the independent variables. In this respect, we price whole life insurance policies
covering individuals of different cohort groups to investigate the relationship of life settlement values to cohort groups. We, then, investigate the premium, interest rate and credit spread sensitivities of life settlements for different cohort groups. We finally discuss the relationship between credit spreads and the stock market performance to understand how life settlement prices may be correlated to stock market.

To price life settlements, we have gathered the following whole life insurance policy quotes from Prudential Financial Inc.:

<table>
<thead>
<tr>
<th>Coverage</th>
<th>Purchase Age</th>
<th>Monthly Premium</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1,000,000</td>
<td>40</td>
<td>$706</td>
</tr>
<tr>
<td>$1,000,000</td>
<td>45</td>
<td>$893</td>
</tr>
<tr>
<td>$1,000,000</td>
<td>50</td>
<td>$1156</td>
</tr>
<tr>
<td>$1,000,000</td>
<td>55</td>
<td>$1469</td>
</tr>
<tr>
<td>$1,000,000</td>
<td>60</td>
<td>$1942</td>
</tr>
<tr>
<td>$1,000,000</td>
<td>65</td>
<td>$2605</td>
</tr>
</tbody>
</table>

For the following analyses, we select the transaction date to be April 09, 2010, the policy purchase age to be 40 and the premium size to be $706 for all cohorts considered.

**Life Settlement Sensitivity to Underlying Distributions**

In pricing life settlements, assumptions are made regarding the distributions of the underlying hazard function, the probability density function and the cumulative distribution function of death to compute the probability of death occurrence at each instant of time and value the cash flows generated by the life settlement. Since these underlying probability distributions are themselves stochastic, they assume different values with corresponding probabilities. Therefore, it is important to analyze how the life settlement prices are distributed with respect to different underlying probability distributions. To analyze this, we run a Monte-Carlo simulation of the different values the underlying distributions can assume and look at the distribution of the life settlement prices with the above assumptions.
The QQ-plot and the normal probability plot suggest that the life settlement prices are normally distributed. To measure the variance of the distribution, we fit a normal distribution to the distribution of life settlement prices using maximum likelihood estimation.

**Definition** Let $\mu_{V_{t_0}}$ and $\sigma_{V_{t_0}}$ be the mean and the standard deviation of the distribution of life settlement prices. Then, $V_{t_0}$ is distributed such that,

$$V_{t_0} \sim N(\$249,970, \$4066)$$  \hspace{1cm} (93)

where the 95% confidence intervals for $\mu_{V_{t_0}}$ and $\sigma_{V_{t_0}}$ are,

$$\mu_{V_{t_0}} \in [\$249,610, \$250,330] \text{ and } \sigma_{V_{t_0}} \in [\$3829, \$4335]$$  \hspace{1cm} (94)

In general, while one can use the expected distributions of the underlying probability distributions to price the life settlement, it is also desirable to determine the distribution of life settlement prices, using a Monte-Carlo simulation methodology, to measure the sensitivity of the life settlement prices to the underlying distributions.
Cohort Sensitivity of Life Settlements

Since the probability of death occurrence at any future age is higher conditional upon survival until a higher age, \( P(D = t_0 + t | D \geq t_0) > P(D = t_0 + t | D \geq t_0) \) for any \( t_0 > t_0 \), it is natural to expect that the secondary market price of the life insurance policy will be higher for individuals who are older at the time of the sale. With higher probability of death occurrence, there is also a higher probability of the secondary market purchaser receiving the terminal payoff, \( V_{\text{terminal}} \), and a lower probability of making the monthly premium payments, \( A_t \).

Consequently, the expected present value of the cash inflow - the terminal payoff \( V_{\text{terminal}} \) - increases while the expected present value of the cash outflow - the premium payments \( A_t \) - decreases, leading to an overall increase in the expected net present value of the cash flows, \( V_t \).

The plot of the life settlement price, \( V_t \), against the cohort of the policy owner, \( C \), for policy purchase age of 40 and monthly premium payment of $706 yields,

As expected, the model predicts that the life settlement price, \( V_t \), quickly goes towards zero as the policy owner’s age at the time of the sale converges to-
On Life Settlement Pricing

Towards the policy purchase age.

**Premium Sensitivity of The Life Settlements**

Since the cash outflow is completely determined by the monthly premium payments, the secondary market price of the life insurance policy is expected to be inversely related to the size of the monthly premium payments. The plot of the life insurance settlement price, $V_{t_0}$, against monthly premium payments, $A_t$, for different cohort groups, $C$, yields,

It appears that the sensitivity of the life settlement price, $V_{t_0}$, to the premium size increases in reverse proportion to the age of the policy owner at the time of the sale. From the model, the sensitivity of the life settlement price to the premium size is simply the partial derivative of the life settlement price with respect to the premium,

\[
\frac{\partial V_{t_0}}{\partial A_t} = \frac{\partial}{\partial A_t} \left\{ -\sum_{i=0}^{\infty} A_t e^{-[R(t_0,t_i)+S_0](t_1-t_0)} \int_{t_i}^{\infty} p_d(f_D(\tau;C|D \geq t_0))d\tau \right\} \tag{95}
\]

\[
= -\sum_{i=0}^{\infty} e^{-[R(t_0,t_i)+S_0](t_1-t_0)} \int_{t_i}^{\infty} p_d(f_D(\tau;C|D \geq t_0))d\tau \tag{96}
\]
Since the \( e^{-[R(t_0,A_t)+S(t_0)][t_1-t_0]} \) term is constant, let \( c_i \) denote it,

\[
\frac{\partial V_{t_0}}{\partial A_{t_0}} = -\sum_{i=0}^{\infty} c_i \int_{t_0}^{\infty} p_{df}(\tau;C) d\tau
\]

(97)

By Lemma 4.7 in Section 4.4,

\[
\frac{\partial V_{t_0}}{\partial A_{t_0}} = -\sum_{i=0}^{\infty} c_i \int_{t_i}^{\infty} \frac{p_{df}(\tau;C)}{1 - cdf(t_0;C)} d\tau
\]

(98)

\[
= -\sum_{i=0}^{\infty} \frac{c_i}{1 - cdf(t_0;C)} \int_{t_i}^{\infty} p_{df}(\tau;C) d\tau
\]

(99)

\[
= -\sum_{i=0}^{\infty} \frac{c_i \cdot (1 - cdf(t_i;C))}{1 - cdf(t_0;C)}
\]

(100)

\[
= -\sum_{i=0}^{\infty} e^{-[R(t_0,A_t)+S(t_0)][t_1-t_0]} \frac{1 - cdf(t_i;C)}{1 - cdf(t_0;C)}
\]

(101)

Since \( cdf(t_i;C) \) is lower for lower ages, \( \frac{\partial V_{t_0}}{\partial A_{t_0}} \) is greater in the absolute sense and therefore a given change in the premium size, \( \Delta A_{t_0} \), leads to a bigger change in the life settlement price \( \Delta V_{t_0} \), which creates the observed convexity of the premium sensitivity plot of life settlements.

**Interest Rate Sensitivity of Life Settlements**

We define the interest rate sensitivity of the life settlement as the change in the life settlement value for a given parallel shift of the entire spot yield curve. For an insurance policy owner born into the 1950 cohort with policy purchase age of 40 and monthly premium payment of $706, the plot of the interest rate sensitivity of the life settlement yields,
The life settlement is highly sensitive to interest rates as the life settlement’s value declines by as much as 7.2% in response to a parallel shift of the yield curve by 50 basis points.

\[ V_{t_0} \text{ is given by,} \]

\[
V_{t_0} = \int_{t_0}^{\infty} V_{\text{terminal}} \cdot e^{-[R(t_0, \tau) + S_{t_0}](\tau - t_0)} \cdot pdf_D(\tau; C | D \geq t_0) d\tau - \sum_{i=0}^{\infty} A_i \cdot e^{-[R(t_0, t_i) + S_{t_0}](t_i - t_0)} \int_{t_i}^{\infty} pdf_D(\tau; C | D \geq t_0) d\tau
\]  

(102)

Then for \( t > t_0 \)

\[
\frac{\partial V_{t_0}}{\partial R(t_0, t)} = -\int_{t_0}^{\infty} V_{\text{terminal}} \cdot e^{-[R(t_0, \tau) + S_{t_0}](\tau - t_0)} \cdot (\tau - t_0) \cdot pdf_D(\tau; C | D \geq t_0) d\tau + \sum_{i=0}^{\infty} A_i \cdot e^{-[R(t_0, t_i) + S_{t_0}](t_i - t_0)} \cdot (t_i - t_0) \int_{t_i}^{\infty} pdf_D(\tau; C | D \geq t_0) d\tau
\]  

(103)

which is the sum of the time weighted present values of the cash flows in the life settlement transaction.
Lemma 7.1 For bonds, Macaulay duration, $D$, is defined as the relative present value weighted time to receive each cash flow,

$$D = \frac{1}{V} \sum_{i=1}^{n} P(i) t(i)$$  \hspace{1cm} (104)

Duration is also a measure of how the value $V$ of a bond changes in relation to parallel shifts of the yield curve,

$$\frac{\partial V}{\partial R} = -D \cdot V$$  \hspace{1cm} (105)

ignore convexity.

Thus, it is possible to define a duration for the life settlement in much the same way. If we let $D^{LS}$ denote the duration of the life settlement,

$$D^{LS} = -\frac{1}{V_{t_0}} \cdot \frac{\partial V_{t_0}}{\partial R_{(t_0, \tau)}}$$  \hspace{1cm} (106)

where $V_{t_0}$ and $\frac{\partial V_{t_0}}{\partial R_{(t_0, \tau)}}$ are defined as shown above. Then,

$$\frac{\Delta V_{t_0}}{V_{t_0}} = -D^{LS} \cdot \Delta R$$  \hspace{1cm} (107)

Corollary to the interest rate sensitivity of the life settlement, the same formula can be used to measure the Z-spread sensitivity of the life settlement. Since the discount factor is composed of both the yield curve and the Z-spread, a change in either one of the terms results in a change of the discount factor.

Lemma 7.2 Let $\psi$ denote the discount factor such that $\psi_{(t_0, t)} = R_{(t_0, t)} + S_{t_0}$.

Then the total differential of $\psi_{(t_0, t)}$ is,

$$d\psi_{(t_0, t)} = dR_{(t_0, t)} + dS_{t_0}$$  \hspace{1cm} (108)

Definition Using the duration concept introduced above, let $D^{GLS}$ denote generalized duration,

$$D^{GLS} = -\frac{1}{V_{t_0}} \cdot \frac{\partial V_{t_0}}{\partial \psi_{(t_0, t)}}$$  \hspace{1cm} (109)

with,

$$\frac{\partial V_{t_0}}{\partial R_{(t_0, t)}} = \frac{\partial V_{t_0}}{\partial \psi_{(t_0, t)}} \cdot \frac{\partial \psi_{(t_0, t)}}{\partial R_{(t_0, t)}} \quad \frac{\partial V_{t_0}}{\partial S_{t_0}} = \frac{\partial V_{t_0}}{\partial \psi_{(t_0, t)}} \cdot \frac{\partial \psi_{(t_0, t)}}{\partial S_{t_0}}$$  \hspace{1cm} (110)
If we apply this new definition to measure the sensitivity of the life settlement to the discount factor,

$$\frac{\partial V_t}{V_t} = -DG_t \cdot \Delta \psi$$

(111)

VIII. Conclusion

As seen from the results section, the life settlement price is highly sensitive to the cohort of the policy owner and to changes in the yield curve. The cohort of the policy owner affects the distribution of the conditional probability density function of time until death and thus determines the weighting of the cash flows. It is therefore important that the modeling of the conditional probability density function of time until death be as robust as possible. Similarly, a model of the yield curve dynamics may be useful in predicting particularly opportune moments to purchase a life settlement. The credit spread model, though not directly applicable to the spot pricing of the life settlement, is helpful in conveying qualitative information about the investors’ credit risk exposure through the life settlement transaction. Also, the implementation of the credit spread model to MetLife Inc. reveals a correlation of 0.5 between the credit spread volatility of MetLife and that of S&P 500, thus challenging the claim by Weber & Hause that a life insurance policy is uncorrelated against almost any other asset class.

The main shortcoming of this paper is its neglect of potential adverse selection. Adverse selection in life settlements may occur due to asymmetric information access between the investor and the policy owner. The policy owner might have information about his health, lifestyle or other factors that may affect his individual conditional probability density function of time until death that the investor might not have access to. It is therefore reasonable to expect that majority of the life insurance policy owners, who create the supply side of the life settlement market, have a certain motivation to sell their policies, other than their financial inability to meet their obligation to make premium payments. In this sense, the life settlement market may resemble the classic lemons market. Taking this information asymmetry into consideration, investors wishing to purchase life settlements should in fact price an additional “information asymmetry premium” into their valuation. Another shortcoming is the limitation on the type of life insurance policy analyzed in this paper. The assumption of continuous whole life insurance policy with fixed premia payments calls for modifications of the model to account for different life insurance contracts and payment patterns.

A technical shortcoming of the model is its assumption of independence
between the Brownian Motions driving the two parameters that describe the exponential distribution of the hazard function. The immediate remedy to this problem seems to be the introduction of correlation into the model to make it more realistic. This expansion of the model, however, is sure to bring additional mathematical complications. Finally, the credit spread model is defined as a mean reverting AR(1) process with a pure stochastic volatility model. While mean-reversion is a reasonable choice, as evidenced by its heavy use in the literature, the pure stochastic volatility model does not provide any economic intuition about the source of the volatility. One may therefore try to regress historical credit spread data on other economic variables such as inflation, growth, stock market performance, etc. to find an economic reasoning for the volatility. Alternatively, one may use a Markov chain to create a regime-switching volatility model (Hamilton, 1989) to account for high and low volatility periods.

Besides technical precision, one should not ignore the role of psychological factors in the development of the life settlement industry. The description of life settlements as “macabre investments” by BusinessWeek (2007) indicates the shocking novelty of the proposed investment vehicle and the unpreparedness of investors for such an instrument. Introduction of death bonds will definitely raise ethical issues and the market may grow at a slower pace than anticipated. However, this shock will eventually wane with the instrument’s continued presence in the capital markets and institutions will incorporate life settlements into their array of investment tools.
References


Human Mortality Database provides detailed mortality and population data to researchers, students, journalists, policy analysts, and others interest in the history of human longevity (www.mortality.org)


Appendix

As mentioned in subsection 6.1 under Assumption 6.3, the residuals are modeled as an Ornstein-Uhlenbeck process with,

\[ \epsilon_{t+dt} = \epsilon_t e^{-\theta dt} + \mu \epsilon (1 - e^{-\theta dt}) + \sigma \epsilon \sqrt{\frac{1 - e^{-2\theta dt}}{2\theta}} X, \quad X \sim N(0, 1) \]  

(112)

Since \( X \) is normally distributed,

\[ \mathbb{E}(\epsilon_t | \epsilon_{t-dt}; \mu \epsilon, \sigma \epsilon, \theta) = \epsilon_t e^{-\theta dt} + \mu \epsilon (1 - e^{-\theta dt}) \]  

(113)

\[ \text{Var}(\epsilon_t | \epsilon_{t-dt}; \mu \epsilon, \sigma \epsilon, \theta) = \sigma^2 \frac{1 - e^{-2\theta dt}}{2\theta} \]  

(114)

Therefore, the conditional probability distribution of \( \epsilon_t \) given \( \epsilon_{t-dt} \) is,

\[ f(\epsilon_t | \epsilon_{t-dt}; \mu \epsilon, \sigma \epsilon, \theta) = \frac{1}{\sqrt{2\pi \sigma^2(1 - e^{-2\theta dt})}} e^{-\frac{(\epsilon_t - \epsilon_{t-dt} e^{-\theta dt} - \mu \epsilon (1 - e^{-\theta dt}))^2}{2\sigma^2(1 - e^{-2\theta dt})}} \]  

(115)

**Definition** Then, the log-likelihood function of a sequence of observations \( (\epsilon_{t_i}, \epsilon_{t_{i+1}}, ..., \epsilon_{t_n}) \), where \( \epsilon_{t_i} - \epsilon_{t_{i-1}} = dt \) is,

\[ \mathcal{L}(\mu \epsilon, \sigma \epsilon, \theta) = \sum_{i=1}^{n} \ln(f(\epsilon_{t_i} | \epsilon_{t_{i-1}}; \mu \epsilon, \sigma \epsilon, \theta)) \]  

(116)

\[ = \frac{n}{2} \ln(2\pi) - n \cdot \ln(\sigma \epsilon \sqrt{\frac{1 - e^{-2\theta dt}}{2\theta}}) \]  

(117)

\[ - \frac{\theta}{\sigma^2(1 - e^{-2\theta dt})} \sum_{i=1}^{n} (\epsilon_{t_i} - \epsilon_{t_{i-1}} e^{-\theta dt} - \mu \epsilon (1 - e^{-\theta dt}))^2 \]

Taking the partial derivatives with respect to each parameter and setting them equal to zero,

\[ \frac{\partial \mathcal{L}(\mu \epsilon, \sigma \epsilon, \theta)}{\partial \mu} = \frac{2\theta}{\sigma^2(1 - e^{-2\theta dt})} \sum_{i=1}^{n} (\epsilon_{t_i} - \epsilon_{t_{i-1}} e^{-\theta dt} - \mu \epsilon (1 - e^{-\theta dt}))) = 0 \]  

(118)

Since \( \theta \neq 0 \), the above expression equals zero only if the sum term equals zero,

\[ \sum_{i=1}^{n} (\epsilon_{t_i} - \epsilon_{t_{i-1}} e^{-\theta dt} - \mu \epsilon (1 - e^{-\theta dt})) = 0 \]  

(119)

Solving the above expression for \( \mu \epsilon \),
To compute for $\sigma_\epsilon$, solving for $\sigma_\epsilon$ yields,

$$
\sigma_\epsilon^2 = \frac{2\vartheta \sum_{i=1}^{n} \left(\epsilon_{t_i} - \mu_e - e^{-\vartheta dt} (\epsilon_{t_{i-1}} - \mu_e)\right)^2}{n(1 - e^{-2\vartheta dt})}
$$  \hfill (122)

Finally, to compute for $\vartheta$, solving the above expression for $\vartheta$,

$$
\vartheta = -\frac{1}{dt} \ln \left\{ \frac{\sum_{i=1}^{n} \left(\epsilon_{t_i} - \mu_e\right) (\epsilon_{t_{i-1}} - \mu_e)}{\sum_{i=1}^{n} \left(\epsilon_{t_{i-1}} - \mu_e\right)^2} \right\}
$$  \hfill (124)

Using algebraic manipulations, the solutions for $\mu_e$, $\sigma_\epsilon$ and $\vartheta$ are,

$$
\mu_e = \frac{\sum_{i=1}^{n} \epsilon_{t_i} \sum_{i=1}^{n} \epsilon_{t_{i-1}}^2 - \sum_{i=1}^{n} \epsilon_{t_{i-1}} \sum_{i=1}^{n} \epsilon_{t_i} \epsilon_{t_{i-1}}^2}{n(\sum_{i=1}^{n} \epsilon_{t_{i-1}}^2 - \sum_{i=1}^{n} \epsilon_{t_{i-1}} \epsilon_{t_i}) - \left( \sum_{i=1}^{n} \epsilon_{t_{i-1}} \right)^2 - \sum_{i=1}^{n} \epsilon_{t_{i-1}} \sum_{i=1}^{n} \epsilon_{t_i}}
$$  \hfill (125)

$$
\sigma_\epsilon = \frac{2\vartheta}{n(1 - e^{-2\vartheta dt})} \left[ \sum_{i=1}^{n} \epsilon_{t_i}^2 - 2\epsilon_{t_i} \sum_{i=1}^{n} \epsilon_{t_{i-1}} + e^{-2\vartheta dt} \sum_{i=1}^{n} \epsilon_{t_i}^2 \right] - 2\mu_e (1 - e^{-\vartheta dt}) \left( \sum_{i=1}^{n} \epsilon_{t_i} - e^{-\vartheta dt} \sum_{i=1}^{n} \epsilon_{t_{i-1}} \right) + n\mu_e^2 (1 - e^{-\vartheta dt})^2
$$  \hfill (126)

$$
\vartheta = -\frac{1}{dt} \ln \left\{ \frac{\sum_{i=1}^{n} \epsilon_{t_{i-1}} \epsilon_{t_i} - \mu_e \sum_{i=1}^{n} \epsilon_{t_{i-1}} - \mu_e \sum_{i=1}^{n} \epsilon_{t_i} + n\mu_e^2}{\sum_{i=1}^{n} \epsilon_{t_{i-1}}^2 - 2\mu_e \sum_{i=1}^{n} \epsilon_{t_{i-1}} + n\mu_e^2} \right\}
$$  \hfill (127)